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LETTER TO THE EDITOR

Lie point-symmetries and Poincaré normal forms for dynamical systems

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Abstract. The problem of finding the extended Lie-point time-independent symmetries of autonomous systems of ordinary differential equations is compared with the Poincaré procedure of reducing the system to linear or normal form, showing a close relationship between the two problems. Some special situations, including the classical Hopf bifurcation problem, are also examined from this point of view.

Let us consider a system of autonomous first-order time-evolution differential equations for the real n-dimensional vector $u = u(t) \in R^n$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(u) \tag{1}$$

where f is an analytical vector field $f \equiv (f_1, f_2, \ldots, f_n)$, $f_i : \Omega \to R^n$ and $\Omega \subset R^n$ is an open neighbourhood of $u = 0$, such that $f(0) = 0$. We shall write, separating the linear part from the remaining higher-order terms

$$f(u) = Lu + h(u) = Lu + h^2(u) + h^{(3)}(u) + \ldots \tag{2}$$

where $h^{(m)}(u)$ is a n-vector field whose components are linear combinations of monomials of degree $m \geq 2$. The problem of finding all Lie-point (LP) symmetries admitted by (1) (in the extended sense, including nonlinear and/or local ones, according to the old idea due to Lie, and recently reconsidered by many authors; see, e.g., [1–4] and references therein) can be stated in the following way. Writing the symmetry generator in the form

$$\eta = \phi_i \partial_i + \tau \partial_t \qquad \partial_i \equiv \partial/\partial u_i, \partial_t \equiv \partial/\partial t \tag{3}$$

where $\phi_i = \phi_i(u, t)$, $\tau = \tau(u, t)$, it can be shown, following [1–4] (see also (5) for further details), that the determining equations for ϕ_i, τ are

$$\phi_i \partial_i f_k - \partial_t \phi_k + f_k \partial_t \tau - f_i \partial_i \phi_k + f_k f_i \partial_i \tau = 0. \tag{4}$$

We are interested here in *time-independent* LP symmetries, so the above conditions become

$$\phi_i \partial_i f_k - f_i \partial_i \phi_k \equiv \{\phi, f\}_k = 0 \tag{5}$$

(having introduced the shorthand notation $\{\,,\}$). Assume now that the functions $\phi_i(u)$ can be constructed as a formal series expansion:

$$\phi_i = \Phi_{ij} u_j + \psi_i^{(2)}(u) + \psi_i^{(3)}(u) + \ldots \tag{6}$$

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where Φ is an $n\times n$ constant matrix, and each $\psi_i^{(m)}(u)$ is a linear combination of monomials of order m. First of all, one immediately deduces from (5) the vanishing of the Lie commutator:

$$[\Phi, L]=0. \tag{7}$$

This reproduces a classical result coming from standard (i.e. linear) theory of equivariant problems [6-8]: we recall that a problem (1) is said to be equivariant with respect to a linear group of transformations G acting on r^n if

$$f(gu)=gf(u) \qquad \forall g\in G. \tag{8}$$

In the case where G is a Lie group of linear transformations, it is easily seen that Φ is the matrix representing the Lie generators of G in the space R^n, and then (7) immediately follows from (8). We can say that (5) constitutes a sort of 'extended (nonlinear) equivariance' of (1) under the action of the group generated by the operators η.

The next step is to write down condition (5) separately for the various orders $m\geqslant 2$

$$\begin{aligned}&(L\psi^{(2)})_k+(Lu)_i\partial_i\psi_k^{(2)}=(\Phi h^{(2)})_k-(\Phi_u)_i\partial_i h_k^{(2)}\\&(L-(Lu)\cdot\partial)\psi^{(3)}=(\Phi-(\Phi u)\cdot\partial)\psi^{(3)}+\{h^{(2)},\psi^{(2)}\}\\&(L-(Lu)\cdot\partial)\psi^{(4)}=(\Phi-(\Phi u)\cdot\partial)\psi^{(4)}+\{h^{(2)},\psi^{(3)}\}+\{h^{(3)},\psi^{(2)}\}\end{aligned} \tag{9}$$

(in the second and third lines a clear shortening of notation with respect to the first one has been adopted), and so on. All equations in (9) have the form of 'homological equations' (cf [9])

$$D_L\psi^{(m)}=w^{(m)}\equiv D_\Phi h^{(m)}+\Sigma_{(m)}\{h^{(a)},\psi^{(b)}\} \tag{9$'$}$$

where at the RHS of each order m the sum is extended to all possible brackets $\{h^{(a)},\psi^{(b)}\}$ giving monomials of degree m, and D_L is the operator

$$D_L=(L-(Lu)_i\partial_i) \tag{9$''$}$$

(and similarly for D_Φ). Then, once a matrix Φ has been chosen in agreement with (7), the RHS of the first line in (9) is known, and we see that if the first p lines of (9) can be solved, then the RHS of the $(p+1)$th equation is also known. Assume now that L can be diagonalized (this is not a restriction, cf [9]), with eigenvalues σ_k and eigenvectors e_k; denoting by $(\tilde u_1,\tilde u_2,\ldots,\tilde u_n)$ coordinates with respect to the basis e_k, then L also is diagonal in the space of vector-valued monomials $u^m e_k$ $(u^m\equiv u_1^{m_1}u_2^{m_2}\ldots u_n^{m_n}$, $m_i\geqslant 0$, $m=m_1+m_2+\ldots+m_n\geqslant 2$; for notational convenience we write simply u instead of $\tilde u$, or assume L diagonal *tout court*), with eigenvalues

$$D_L u^m e_k=(\sigma_k-(\sigma, m))u^m e_k \tag{10}$$

where $(\sigma, m)=\sigma_1 m_1+\ldots+\sigma_n m_n$, and the mth equation of the system (9) splits into $N=\binom{m+m-1}{m}$ equations, N being the number of all monomials of degree m. Then, each one of these equations can be solved if the RHS of (10) is never zero, i.e. if all the eigenvalues σ_k of L are non-resonant. But this is precisely the same condition ensuring the solvability of each step of the classical Poincaré method for reducing the system (1) to its linear part [9]. Therefore, we can say:

Proposition 1. The sufficient conditions on the eigenvalues of the linear part L of (1) which ensure, according to the Poincaré procedure, the reduction of (1) to the linear form by a formal (respectively converging) series, ensure also the existence of a LP symmetry written as a formal (respectively converging) series.

The main difference here with respect to the Poincaré procedure for reduction to linear or normal forms is the special form of the RHS of all equations (9). More precisely, the arbitrariness in the matrix Φ (only condition (7) is to be satisfied) may allow us to solve (9), even in the presence of resonances, by choosing Φ, whenever possible, in such a way that the RHS of all equations in (9) which contain resonant eigenvalues is also equal to zero. In particular, concerning the problem of finding LP symmetries of (1), some further considerations can be of interest (see also [5] for futher detail). First of all, it can be noted that in the algebra generated by the LP symmetries of (1), there is always an 'obvious' generator, satisfying (5), which is given by

$$\eta f = f_i \partial_i \tag{11}$$

(i.e. $\phi_i = f_i$, $\psi^{(k)} = h^{(k)}$) which corresponds to the time evolution of solutions: $u(t) \rightarrow u(t+\varepsilon)$ (in fact, $\partial_t - f_i \partial_i \equiv 0$ on each solution). Let us remark also that if η is a symmetry generator of (1), the same is also true for

$$\eta' = y(u, t)\eta \tag{12}$$

if y is any solution of the linear PDE

$$\partial_t y + f_i \partial_i y = 0.$$

This means that y is a 'constant of motion' along any solution of (1), and η and η' have the same effect once applied to any solution $u(t)$ of (1): this allows us to consider η and η' as 'identical symmetries' and, when enumerating the possible independent symmetries, to look only for truly different symmetries. Some simple consequences of (9) in some special cases are presented in the following propositions.

Proposition 2. If all the eigenvalues σ_k of L are distinct and non-resonant, then we may choose $\Phi^{(k)} = P^{(k)}$, the orthogonal projection operator on the eigenvector e_k of L, and then construct through (9) n linearly independent symmetries $\eta^{(k)}$ of (1). In particular, in the case $h = 0$, i.e. if the problem (1) is linear (or reduced to this form), these symmetries are given by $\eta^{(k)} = u_k \partial_k$ (no sum over k), i.e. by dilations along the direction e_k.

Let us remark that the above symmetries for the linear case form an Abelian algebra. Notice also that the linear case admits always symmetries of the form $\eta_\alpha = \alpha_k \partial_k$, where $\alpha = \alpha(t)$ is any solution of the given system, a fact which corresponds to the linear superposition property; in any case, they do not enter in our discussion, since we limited it to time-independent symmetries.

Proposition 3. Suppose that the nonlinear part $h(u)$ in (1) contains just one monomial $u^m e_k$, without any hypothesis on the presence of resonances. Then there are $(n-1)$ linear symmetries for the system (1).

Proof. All equations not containing $u^m e_k$ can be solved by $\psi^{(s)} = 0$, and the remaining term $\psi_k^{(m)}$ proportional to $u^m e_k$ is determined by the equation

$$(\sigma_k - (\sigma, m))\psi_k^{(m)} = D_\Phi u^m e_k.$$

Putting

$$\Phi = \mathrm{diag}(\rho_1, \ldots, \rho_n) \tag{13}$$

this equation can be solved by choosing ρ_i in such a way that $\rho_k = (\rho, m)$ and $\psi_k^{(m)} = 0$, and this gives $n-1$ independent linear symmetries, whether or not $u^m e_k$ is a resonant term. The nth possible choice $\rho_i = \sigma_i$ and $\psi_k^{(m)} \neq 0$ produces the symmetry $\eta = \eta_f$.

Proposition 4. If all nonlinear terms in the system (1) are resonant, there is at least one linear symmetry, which generates the scaling $u_i \to u_i \exp(\sigma_i \varepsilon)$, $\varepsilon \in R$.

Proof. All equations (9) can be solved by $\psi_i^{(k)} = 0$ for all k, i, and choosing the elements ρ_i of Φ (in the form (13)) satisfying all conditions $\rho_k = (\rho, m)$ for all monomials $u^m e_k$ appearing in the nonlinear part of the given system (1). The number of independent solutions ρ_i of all these conditions gives the number of the possible linear symmetries admitted by (1); the hypothesis that all the appearing monomials are resonant ensures that at least the choice $\sigma_i = \rho_i$ is a solution.

The last proposition is interesting because it is known that, according to the Poincaré–Dulac theorem [9], any system (1) may be converted, by a formal or converging series, into a system containing only resonant terms.

A particular case is given when $n = 2$ and the eigenvalues of L are imaginary, say $\pm i\omega$, and then resonant, as in the standard Hopf bifurcation problem [6–8]. Once reduced to the normal form, the linear symmetry generated according to proposition 4 can be written in the form $z \to z \exp(i\omega\varepsilon)$, having introduced the complex vector $z = u_1 + iu_2$ as usual: then this expresses just the known property that the normal form of this problem exhibits an explicit equivariance under the rotation group $S^1 = SO_2$ [6–8].

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